J. Appl. Prob. **54,** 763–772 (2017)
*doi:*10.1017/jpr.2017.33

ALMOST STOCHASTIC DOMINANCE UNDER INCONSISTENT UTILITY AND LOSS FUNCTIONS

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Abstract

Current literature on stochastic dominance assumes utility/loss functions to be the same across random variables. However, decision models with inconsistent utility functions have been proposed in the literature. The use of inconsistent loss functions when comparing between two random variables can also be appropriate under other problem settings. In this paper we generalize almost stochastic dominance to problems with inconsistent utility/loss functions. In particular, we propose a set of conditions that is necessary and sufficient for clear preferences when the utility/loss functions are allowed to vary across different random variables.

Keywords: Stochastic dominance; almost stochastic dominance; utility; loss; probability distribution

2010 Mathematics Subject Classification: Primary 60E15
Secondary 90B50; 91B16

1. Introduction

Stochastic dominance is a form of stochastic ordering that is frequently used in decision analysis to determine if a clear preference exists between two distributions of outcomes. For example, a random variable X dominates another random variable Y with first-degree stochastic dominance if and only if the expected utility of X is at least as large as the expected utility of Y across all nondecreasing utility functions. Hence, stochastic dominance can reveal preferences even when the utility function of the decision maker is unclear. There are many forms of stochastic ordering, including the hazard rate order and inverse stochastic dominance which were considered in [20] and [5], respectively. We refer the interested reader to [9], [10], [12], and [15] for a survey on this topic.

In the stochastic dominance literature, the expected utility associated with the two random variables under consideration is often based on the same utility function. However, these stochastic dominance rules are not applicable to decision models that allow for inconsistent utility functions. For example, a decision model where utility is lottery dependent was proposed in [1] and subsequently considered in [2], [4], and [14].

In the quality engineering literature, expected loss is frequently used to compare different distributions of outcomes. Similar to expected utility, expected loss is computed using the same loss function across all random variables under consideration. However, this may not be

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appropriate under certain problem settings. For example, the concentration levels of chemical compounds that are known to have adverse effects on the ecology and/or human health are constantly monitored and the loss associated with individual chemical compounds is frequently expressed as a function of observed concentration levels [3], [6], [7]. However, the relationship between loss and concentration is often poorly understood and likely to differ for different chemical compounds in practice.

In this paper we build on the concept of almost stochastic dominance (ASD) proposed in [8], which is able to identify clear preferences in practice that are not revealed by conventional stochastic dominance rules. For example, most investors with a sufficiently long investment horizon will prefer stocks over bonds and the former dominates the latter with ASD, but not under conventional stochastic dominance rules. Recent generalizations of ASD include ASD of higher degrees [19], generalized ASD [18], weighted ASD [16], and between first- and second-order stochastic dominance [11]. In the above works, the utility/loss function was assumed to be the same across all random variables.

We address this limitation by proposing a set of conditions that is necessary and sufficient for clear preferences when the utility/loss functions associated with different distributions of outcomes are allowed to vary. We illustrate how our proposed conditions, which generalize the ASD conditions proposed in [8], can be used through an example of comparing poorly understood chemical compounds that are present in the environment.

2. Main results

Consider two random variables X and Y. Let F and G denote the cumulative distribution function of X and Y, respectively. Let u_X and u_Y denote the utility function associated with X and Y, respectively. Without loss of generality, assume that $\mathbb{E}[X] \geq \mathbb{E}[Y]$. In addition, we assume that u_X and u_Y are nondecreasing (i.e. $u'_X, u'_Y \geq 0$) and their marginal utilities are bounded from above and below by $\bar{u}'$ and $\underline{u}'$, respectively, i.e.

$$\bar{u}' = \sup\{u'_X, u'_Y\} \quad \text{and} \quad \underline{u}' = \inf\{u'_X, u'_Y\}.$$

Furthermore, u_X and u_Y are known to be equivalent at k distinct points $t_1, t_2, \dots, t_k$, i.e.

$$u_X(t_1) = u_Y(t_1), \qquad u_X(t_2) = u_Y(t_2), \dots, u_X(t_k) = u_Y(t_k).$$

Without loss of generality, assume that $t_1 < t_2 < \cdots < t_k$. Here, we note that our problem generalizes the problem that was studied in [8]. In particular, our problem reduces to the latter when $u_X(t) = u_Y(t)$ for all t.

Definition 2.1. Define tolerance τ as

$$\tau = \frac{\int_{t_k}^{\infty}(1-F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t + \sum_{i=1}^{k-1}\int_{t_i^*}^{t_{i+1}}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t}{\int_{t_k}^{\infty}(1-G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t},$$

where

$$t_i^* = \begin{cases} \max\{t \in [t_i, t_{i+1}] \colon G(t) \leq F(t_i + t_{i+1} - t)\} & \text{if } F(t_{i+1}) \geq G(t_i) \\ t_i & \text{otherwise.} \end{cases} \tag{2.1}$$

Proposition 2.1. *If* $\mathbb{E}[X] \geq \mathbb{E}[Y]$ *then* $1 \leq \tau \leq \int_{\bar{S}_1}[G(t) - F(t)]\,\mathrm{d}t / \int_{S_1}[F(t) - G(t)]\,\mathrm{d}t$, *where* $S_1 = \{t : F(t) > G(t)\}$ *and* $\bar{S}_1$ *denotes the complement of* S_1.

Proof. First, we show that $\tau \geq 1$. By Definition 2.1,

$$\begin{aligned}
\tau &= \frac{\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t + \sum_{i=1}^{k-1}\int_{t_i^*}^{t_{i+1}}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t}{\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t} \\
&= 1 + \frac{\int_{-\infty}^{\infty} G(t)\,\mathrm{d}t - \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \int_{t_k}^{\infty} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} F(t_i + t_{i+1} - t)\,\mathrm{d}t}{\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t} \\
&= 1 + \frac{\int_{-\infty}^{\infty} G(t)\,\mathrm{d}t - \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \int_{t_k}^{\infty} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} F(s)\,\mathrm{d}s}{\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t} \qquad (2.2) \\
&= 1 + \frac{\int_{-\infty}^{\infty} G(t) - F(t)\,\mathrm{d}t}{\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t} \\
&= 1 + \frac{\mathbb{E}[X] - \mathbb{E}[Y]}{\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t} \qquad (2.3) \\
&\geq 1. \qquad (2.4)
\end{aligned}$$

Equation (2.2) follows from the substitution $s = t_i + t_{i+1} - t$. Equation (2.3) follows from the definition of expected value and by applying integration by parts. Equation (2.4) follows from the fact that $\mathbb{E}[X] \geq \mathbb{E}[Y]$, $G(t) \in [0, 1]$, $F(t) \in [0, 1]$, and $\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t \leq 0$; see (2.1). Hence, $\tau \geq 1$.

Next, we show that $\tau \leq \int_{\bar{S}_1}[G(t) - F(t)]\,\mathrm{d}t / \int_{S_1}[F(t) - G(t)]\,\mathrm{d}t$. First, we highlight the fact that $\tau = \int_{\bar{S}_1}[G(t) - F(t)]\,\mathrm{d}t / \int_{S_1}[F(t) - G(t)]\,\mathrm{d}t$ if $u_X(t) = u_Y(t)$ for all t. Next, we note that $\tau(t_1, t_2, t_{i-1}, t_{i+1}, \ldots, t_k) \leq \tau(t_1, t_2, \ldots, t_k)$, where $\tau(t_1, t_2, \ldots, t_k)$ denotes the value of τ when u_X and u_Y are known to be equivalent at k distinct points $t_1, t_2, \ldots, t_k$. This follows from (2.3), which states that

$$\tau = 1 + \frac{\mathbb{E}[X] - \mathbb{E}[Y]}{\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_i^*}(G(t) - F(t_i + t_{i+1} - t))\,\mathrm{d}t}$$

and (2.1), which highlights that the third term in the denominator of the expression above is nondecreasing when the equivalent point t_i is removed.

Since $\tau = \int_{\bar{S}_1}[G(t) - F(t)]\,\mathrm{d}t / \int_{S_1}[F(t) - G(t)]\,\mathrm{d}t$ when $u_X(t) = u_Y(t)$ for all t and τ is nonincreasing with the removal of an equivalent point, $\tau \leq \int_{\bar{S}_1}[G(t) - F(t)]\,\mathrm{d}t / \int_{S_1}[F(t) - G(t)]\,\mathrm{d}t$. □

Proposition 2.1 states that τ is some value between 1 and the ratios of the area between F and G (i.e. $\int_{\bar{S}_1}[G(t) - F(t)]\,\mathrm{d}t / \int_{S_1}[F(t) - G(t)]\,\mathrm{d}t$). Next, we present a theorem which highlights that τ describes the maximum allowable deviation in marginal utility such that preference for X over Y is clear. In particular, the expected utility of X is no less than the expected utility of Y if their marginal utilities deviate by a maximum factor of τ. Furthermore, if their marginal utilities are allowed to deviate by a factor greater than τ, there exist some u_X and u_Y such that the expected utility of X is strictly less than the expected utility of Y.

Theorem 2.1. *Suppose that* $u_X(t_1) = u_Y(t_1), u_X(t_2) = u_Y(t_2), \ldots, u_X(t_k) = u_Y(t_k)$, *and* $\mathbb{E}[X] \geq \mathbb{E}[Y]$. *Then* $\mathbb{E}[u_X(X)] \geq \mathbb{E}[u_Y(Y)]$ *if and only if* $\sup\{u'_X, u'_Y\}/\inf\{u'_X, u'_Y\} \leq \tau$, *where* τ *is defined by Definition 2.1.*

Proof. First, we show that if $\sup\{u'_X, u'_Y\}/\inf\{u'_X, u'_Y\} \leq \tau$ then $\mathbb{E}[u_X(X)] \geq \mathbb{E}[u_Y(Y)]$. We have

$$
\begin{aligned}
&\mathbb{E}[u_X(X)] - \mathbb{E}[u_Y(Y)] \\
&\quad = \int_{-\infty}^{\infty} u_X(t)\,\mathrm{d}F(t) - \int_{-\infty}^{\infty} u_Y(t)\,\mathrm{d}G(t) \\
&\quad = \bigg[\bigg(u_X(t_k) + \int_{t_k}^{\infty} u'_X(t)\,\mathrm{d}t\bigg) - \int_{-\infty}^{\infty} u'_X(t)F(t)\,\mathrm{d}t\bigg] \\
&\qquad - \bigg[\bigg(u_Y(t_k) + \int_{t_k}^{\infty} u'_Y(t)\,\mathrm{d}t\bigg) - \int_{-\infty}^{\infty} u'_Y(t)G(t)\,\mathrm{d}t\bigg] \\
&\quad = \bigg[\int_{t_k}^{\infty} u'_X(t)(1 - F(t))\,\mathrm{d}t - \int_{-\infty}^{t_1} u'_X(t)F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} u'_X(t)F(t)\,\mathrm{d}t\bigg] \\
&\qquad - \bigg[\int_{t_k}^{\infty} u'_Y(t)(1 - G(t))\,\mathrm{d}t - \int_{-\infty}^{t_1} u'_Y(t)G(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} u'_Y(t)G(t)\,\mathrm{d}t\bigg] \qquad (2.5) \\
&\quad \geq \underline{u}'\bigg(\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t\bigg) - \bar{u}'\bigg(\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t\bigg) \\
&\qquad + \sum_{i=1}^{k-1}\bigg(\int_{t_i}^{t_{i+1}} u'_Y(t)G(t)\,\mathrm{d}t - \int_{t_i}^{t_{i+1}} u'_X(t)F(t)\,\mathrm{d}t\bigg) \qquad (2.6) \\
&\quad = \underline{u}'\bigg(\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t\bigg) - \bar{u}'\bigg(\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t\bigg) \\
&\qquad + \sum_{i=1}^{k-1}\bigg(\int_{t_i}^{t_{i+1}} u'_Y(t)G(t)\,\mathrm{d}t - \int_{t_i}^{t_{i+1}} u'_X(t_i + t_{i+1} - s)F(t_i + t_{i+1} - s)\,\mathrm{d}s\bigg) \qquad (2.7) \\
&\quad = \underline{u}'\bigg(\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t\bigg) - \bar{u}'\bigg(\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t\bigg) \\
&\qquad + \sum_{i=1}^{k-1}\bigg(\int_{t_i}^{t_{i+1}} u'_Y(t)G(t)\,\mathrm{d}t - u'_X(t_i + t_{i+1} - t)F(t_i + t_{i+1} - t)\,\mathrm{d}t\bigg) \\
&\quad \geq \underline{u}'\bigg(\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t\bigg) - \bar{u}'\bigg(\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t\bigg) \\
&\qquad + \sum_{i=1}^{k-1}\bigg(\bar{u}'\int_{t_i}^{t_i^*} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t + \underline{u}'\int_{t_i^*}^{t_{i+1}} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t\bigg) \\
&\qquad\qquad (2.8) \\
&\quad = \underline{u}'\bigg(\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t + \sum_{i=1}^{k-1}\int_{t_i^*}^{t_{i+1}} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t\bigg)
\end{aligned}
$$

$$- \bar{u}'\bigg(\int_{t_k}^{\infty} (1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1} \int_{t_i}^{t_i^*} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t\bigg)$$
$$\geq 0. \tag{2.9}$$

Equation (2.5) follows from the fact that $u_X(t_k) = u_Y(t_k)$. Equation (2.6) follows from the fact that $\underline{u}' \leq u_X' \leq \bar{u}'$ and $\underline{u}' \leq u_Y' \leq \bar{u}'$. Equation (2.7) follows from the substitution $t = t_i + t_{i+1} - s$ and (2.8) follows from Lemma A.1; see Appendix A. Equation (2.9) follows from Definition 2.1 and the condition $\sup\{u_X', u_Y'\}/\inf\{u_X', u_Y'\} \leq \tau$. Hence, it follows from above that $\mathbb{E}[u_X(X)] \geq \mathbb{E}[u_Y(Y)]$ if $\sup\{u_X', u_Y'\}/\inf\{u_X', u_Y'\} \leq \tau$.

Next we prove the other direction by contradiction. Assume that

$$\mathbb{E}[u_X(X)] \geq \mathbb{E}[u_Y(Y)], \tag{2.10}$$

and

$$\frac{\sup\{u_X', u_Y'\}}{\inf\{u_X', u_Y'\}} > \tau. \tag{2.11}$$

Let $\gamma = \sup\{u_X', u_Y'\}/\inf\{u_X', u_Y'\}$. The following u_X and u_Y are consistent with assumption (2.11):

$$u_X'(t) = \begin{cases} \gamma^{0.5}, & t \in [-\infty, t_1] \cup [t_i + t_{i+1} - t_i^*, t_{i+1}],\ i = 1, 2, \ldots, k-1, \\ \dfrac{1}{\gamma^{0.5}}, & t \in [t_i, t_i + t_{i+1} - t_i^*] \cup [t_k, \infty],\ i = 1, 2, \ldots, k-1, \end{cases} \tag{2.12}$$

$$u_Y'(t) = \begin{cases} \dfrac{1}{\gamma^{0.5}}, & x \in [-\infty, t_1] \cup [t_i^*, t_{i+1}],\ i = 1, 2, \ldots, k-1, \\ \gamma^{0.5}, & x \in [t_i, t_i^*] \cup [t_k, \infty],\ i = 1, 2, \ldots, k-1. \end{cases} \tag{2.13}$$

It follows that

$$\begin{aligned}
&\mathbb{E}[u_X(X)] - \mathbb{E}[u_Y(Y)] \\
&\quad = \bigg[\int_{t_k}^{\infty} u_X'(t)(1 - F(t))\,\mathrm{d}t - \int_{-\infty}^{t_1} u_X'(t)F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} u_X'(t)F(t)\,\mathrm{d}t\bigg] \\
&\quad\quad - \bigg[\int_{t_k}^{\infty} u_Y'(t)(1 - G(t))\,\mathrm{d}t - \int_{-\infty}^{t_1} u_Y'(t)G(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} u_Y'(t)G(t)\,\mathrm{d}t\bigg] \qquad (2.14) \\
&\quad = \frac{1}{\gamma^{0.5}}\bigg(\int_{t_k}^{\infty}(1 - F(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t\bigg) - \gamma^{0.5}\bigg(\int_{t_k}^{\infty}(1 - G(t))\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t\bigg) \\
&\quad\quad + \sum_{i=1}^{k-1}\bigg(\int_{t_i}^{t_i^*} \gamma^{0.5} G(t)\,\mathrm{d}t + \int_{t_i^*}^{t_{i+1}} \frac{1}{\gamma^{0.5}} G(t)\,\mathrm{d}t - \int_{t_i}^{t_i + t_{i+1} - t_i^*} \frac{1}{\gamma^{0.5}} F(t)\,\mathrm{d}t \\
&\quad\quad\quad - \int_{t_i + t_{i+1} - t_i^*}^{t_{i+1}} \gamma^{0.5} F(t)\,\mathrm{d}t\bigg) \qquad (2.15) \\
&\quad = \frac{1}{\gamma^{0.5}}\bigg[\int_{t_k}^{\infty} 1 - F(t)\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t + \sum_{i=1}^{k-1}\bigg(\int_{t_i^*}^{t_{i+1}} G(t)\,\mathrm{d}t - \int_{t_i}^{t_i + t_{i+1} - t_i^*} F(t)\,\mathrm{d}t\bigg)\bigg]
\end{aligned}$$

$$
\begin{aligned}
&- \gamma^{0.5}\bigg[\int_{t_k}^{\infty} 1 - G(t)\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\bigg(\int_{t_i}^{t_i^*} G(t)\,\mathrm{d}t - \int_{t_i+t_{i+1}-t_i^*}^{t_{i+1}} F(t)\,\mathrm{d}t\bigg)\bigg] \\
&= \frac{1}{\gamma^{0.5}}\bigg[\int_{t_k}^{\infty} 1 - F(t)\,\mathrm{d}t + \int_{-\infty}^{t_1} G(t)\,\mathrm{d}t + \sum_{i=1}^{k-1}\bigg(\int_{t_i^*}^{t_{i+1}} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t\bigg)\bigg] \\
&\quad - \gamma^{0.5}\bigg[\int_{t_k}^{\infty} 1 - G(t)\,\mathrm{d}t + \int_{-\infty}^{t_1} F(t)\,\mathrm{d}t - \sum_{i=1}^{k-1}\bigg(\int_{t_i}^{t_i^*} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t\bigg)\bigg] \qquad (2.16) \\
&< 0. \qquad (2.17)
\end{aligned}
$$

Equation (2.14) follows from (2.5). Equation (2.15) follows from (2.12) and (2.13). Equation (2.16) follows by applying a substitution similar to that used to obtain (2.7). Equation (2.17), which follows from Definition 2.1 and (2.11), contradicts assumption (2.10). This completes the proof. □

Theorem 2.1 states that tolerance τ describes the maximum allowable deviation in marginal utility such that preference for X over Y is clear. Next, we study two problem settings where τ is interesting.

First, we consider the case where X and Y are bounded from below by 0, and u_X and u_Y are known to be equivalent at that lower bound.

Proposition 2.2. *If* $\mathbb{E}[X] \geq \mathbb{E}[Y]$, $u_X(0) = u_Y(0)$, *and X and Y are bounded from below by 0, then* $\tau = \mathbb{E}[X]/\mathbb{E}[Y]$.

Proof. It follows from Definition 2.1 that under the proposition conditions

$$
\tau = \frac{\int_0^{\infty} 1 - F(t)\,\mathrm{d}t}{\int_0^{\infty} 1 - G(t)\,\mathrm{d}t} = \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}.
$$
□

In Proposition 2.2 we highlighted that tolerance is given by the ratio of expected values when both random variables are bounded from below by 0 and the two utility functions are known to be equivalent at that lower bound. Hence, the higher the relative difference in expected values, the higher the allowable deviation in marginal utility for clear preferences. We note that Tan and Luo [17] recently highlighted a similar observation between two random variables under limited distribution information (i.e. only the expected values are known) when $u_X(t) = u_Y(t)$ for all t.

Next, we consider the case where both random variables are bounded from below and above by 0 and 1, respectively. In addition, u_X and u_Y are known to be equivalent at both 0 and 1. In particular, these assumptions are consistent with the problems considered in [1] and [2].

Proposition 2.3. *If* $\mathbb{E}[X] \geq \mathbb{E}[Y]$, $u_X(0) = u_Y(0)$, $u_X(1) = u_Y(1)$, *and X and Y are bounded from below by 0 and bounded from above by 1, then*

$$
\tau = \frac{\int_{t_1^*}^{1} G(t) - F(1-t)\,\mathrm{d}t}{\int_0^{t_1^*} F(1-t) - G(t)\,\mathrm{d}t},
$$

where $t_1^* = \max\{t \in [0, 1] \colon G(t) \leq F(1-t)\}$.

Proof. This follows immediately from Definition 2.1. □

When $u_X(t) = u_Y(t)$ for all t, Leshno and Levy [8] highlighted that tolerance depends on the areas sandwiched between $F(t)$ and $G(t)$. When u_X and u_Y can differ, but are known to be equivalent at the lower and upper bounds, it follows from Proposition 2.3 that tolerance depends on the areas sandwiched between $F(1-t)$ and $G(t)$.

Example 2.1. (*Comparing chemical compounds.*) Let B and C denote the concentration levels (ng/L) of Bisphenol A (BPA) and caffeine in the environment, respectively. Suppose that based on historical data, it is reasonable to assume that B and C are well described by the following triangular distributions:

$$B \sim \text{tri}(0, 360, 48) \quad \text{and} \quad C \sim \text{tri}(0, 2080, 260).$$

As the adverse health effects of BPA are more severe than that of caffeine, the loss of BPA will be higher than that of caffeine under similar concentration levels. To address such differences, some form of normalization is necessary. One way is to divide the concentration of each compound by their respective predicted no-effect concentration (PNEC) values [6], [3]. The proposed PNEC values for BPA and caffeine are 60 ng/L and 5200 ng/L, respectively [13], [7].

Let X and Y denote the 'normalized concentration levels' of BPA and caffeine, respectively, i.e.

$$X \sim \text{tri}(0, 6, 0.8) \quad \text{and} \quad Y \sim \text{tri}(0, 0.4, 0.05).$$

Let l_X and l_Y denote the loss associated with BPA and caffeine, respectively. Here, we assume that $l_X(0) = l_Y(0)$ and $l_X(1) = l_Y(1)$, but it is unclear if the two loss functions are equivalent at other points since the normalization is only carried out at one point (i.e. PNEC value).

It follows from Theorem 2.1 that the expected loss of BPA is no less than the expected loss of caffeine so long as the maximum marginal loss and minimum marginal loss of both compounds differ by less than a factor of 273.577. For example, consider the following loss functions, which are equivalent at 0 and 1 and whose marginal loss differs by less than a factor of 273.577:

$$l_X(x) = \begin{cases} 0.02x, & 0 \le x \le \frac{11}{60}, \\ 1.22x - 0.22, & \frac{11}{60} \le x \le 1, \\ 1.5x - 0.5, & x > 1, \end{cases} \qquad l_Y(y) = \begin{cases} 0.8y, & 0 \le y \le 0.75, \\ 1.6y - 0.6, & 0.75 \le y \le 1, \\ 2y - 1, & y > 1. \end{cases}$$

Based on the loss functions above, $\mathbb{E}[l_X(X)] = 2.92 > \mathbb{E}[l_Y(Y)] = 0.12$, which is consistent with Theorem 2.1. Hence, it can be concluded that based on historical data, the expected loss of BPA is guaranteed to be higher than that of caffeine across a very wide range of loss functions and a decision to prioritize BPA over caffeine is appropriate even though the relationship between loss and concentration is not well understood.

For the sake of completeness, we highlight that Theorem 2.1 also states that the expected loss of caffeine can be strictly greater than the expected loss of BPA if the maximum marginal loss and minimum marginal loss of both compounds are allowed to differ by more than a factor of 273.577. To see this, consider the following loss functions:

$$l_X(x) = \begin{cases} 0.05x, & 0 \le x \le 0.9404, \\ 16x - 15, & 0.9404 \le x \le 1, \\ 0.06x + 0.94, & x > 1, \end{cases}$$

$$l_Y(y) = \begin{cases} 15y, & 0 \le y \le 0.0642, \\ 0.04y + 0.96, & 0.0642 \le y \le 1, \\ 6y - 5, & y > 1. \end{cases}$$

Based on the loss functions above, $\mathbb{E}[l_X(X)] = 0.897 < \mathbb{E}[l_Y(Y)] = 0.90$, which is consistent with Theorem 2.1.

3. Summary and future work

In the stochastic dominance literature, random variables are compared based on the same utility function. These conditions do not apply to decision models with inconsistent utility functions that were motivated by empirically observed violations of expected utility theory and considered in, for example, [1], [2], [4], and [14]. As illustrated in Example 2.1, inconsistent loss functions can also be more appropriate for the comparison of poorly understood chemical compounds where the normalization of loss functions is only performed at selected concentration points. In this paper we generalized the ASD conditions proposed in [8] by providing a set of necessary and sufficient conditions for clear preferences between distributions of outcomes when utility/loss functions are allowed to differ. The conditions highlighted that expected utility/loss of one random variable is guaranteed to be no less than that of another random variable if and only if marginal utilities/losses deviate by less than a factor of τ, which is defined based on the distributions of the random variables under comparison.

In this paper it was assumed that utility/loss functions are only known to be nondecreasing and equivalent at some distinct points. In practice, more information on utility/loss functions may be available (e.g. concavity). One direction of future work is to study how additional information on the properties of the utility/loss functions can be incorporated.

Appendix A

Lemma A.1. *The following inequality holds for* $i = 1, 2, \dots, k-1$*:*

$$\int_{t_i}^{t_{i+1}} u_Y'(t)G(t) - u_X'(t_i + t_{i+1} - t)F(t_i + t_{i+1} - t)\,\mathrm{d}t \geq \bar{u}' \int_{t_i}^{t_i^*} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t + \underline{u}' \int_{t_i^*}^{t_{i+1}} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t.$$

Proof. Let

$$v'(t) = \begin{cases} \bar{u}' & \text{if } t \in [t_i, t_i + m],\ i = 1, 2, \dots, k-1, \\ \underline{u}' & \text{if } t \in (t_i + m, t_{i+1}],\ i = 1, 2, \dots, k-1, \end{cases}$$

where $m = (u_Y(t_{i+1}) - u_Y(t_i) - \underline{u}'(t_{i+1} - t_i))/(\bar{u}' - \underline{u}')$. Since $G(t)$ is nondecreasing

$$\int_{t_i}^{t_{i+1}} u_Y'(t)G(t)\,\mathrm{d}t \geq \int_{t_i}^{t_{i+1}} v'(t)G(t)\,\mathrm{d}t = \bar{u}' \int_{t_i}^{t_i+m} G(t)\,\mathrm{d}t + \underline{u}' \int_{t_i+m}^{t_{i+1}} G(t)\,\mathrm{d}t. \tag{A.1}$$

In a similar fashion, it can be shown that

$$\int_{t_i}^{t_{i+1}} u_X'(t_i + t_{i+1} - t)F(t_i + t_{i+1} - t)\,\mathrm{d}t \leq \bar{u}' \int_{t_i}^{t_i+m} F(t_i + t_{i+1} - t)\,\mathrm{d}t + \underline{u}' \int_{t_i+m}^{t_{i+1}} F(t_i + t_{i+1} - t)\,\mathrm{d}t. \tag{A.2}$$

Hence, from (A.1) and (A.2), it follows that

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} u_Y'(t)G(t) - u_X'(t_i + t_{i+1} - t)F(t_i + t_{i+1} - t)\,\mathrm{d}t \\
&\quad \geq \bar{u}' \int_{t_i}^{t_i+m} G(t)\,\mathrm{d}t + \underline{u}' \int_{t_i+m}^{t_{i+1}} G(t)\,\mathrm{d}t - \bar{u}' \int_{t_i}^{t_i+m} F(t_i + t_{i+1} - t)\,\mathrm{d}t \\
&\qquad - \underline{u}' \int_{t_i+m}^{t_{i+1}} F(t_i + t_{i+1} - t)\,\mathrm{d}t \\
&\quad = \bar{u}' \int_{t_i}^{t_i+m} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t + \underline{u}' \int_{t_i+m}^{t_{i+1}} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t \\
&\quad \geq \bar{u}' \int_{t_i}^{t_i^*} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t + \underline{u}' \int_{t_i^*}^{t_{i+1}} G(t) - F(t_i + t_{i+1} - t)\,\mathrm{d}t
\end{aligned}$$

The last inequality follows from (2.1). □

Acknowledgements

The authors thank the editor, an associate editor, and two anonymous referees for many helpful comments and suggestions. In addition, the authors thank Karina Gin and Luhua You for introducing the chemical compound comparison problem which motivated this work. This research is supported by the National Research Foundation, Prime Ministers Office, Singapore under its Campus for Research Excellence and Technological Enterprise (CREATE) programme.

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