Dominance of the Maximum Geometric Mean Portfolio in the Long Run

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Abstract

It follows from the law of large numbers that the maximum geometric mean (MGM) portfolio will almost surely outperform other portfolios in the long run under mild conditions. However, from a theoretical perspective, preference for the MGM portfolio is not immediately clear as there exist non-decreasing utility functions where a MGM portfolio does not maximize expected utility, even in the long run. In this paper, we show that the MGM portfolio is preferred by most decision makers under a sufficiently long investment horizon by highlighting that the expected utility of the MGM portfolio is higher than the expected utility of all other portfolios for utility functions that describe the preferences of most investors in practice.

Keywords
weighted almost stochastic dominance, maximum geometric mean, mean-variance.

1. Introduction

It has been proposed by many that an investor with a sufficiently long horizon should adopt a maximum geometric mean (MGM) strategy, which aims for maximal terminal wealth by investing in each period based on the logarithm of returns (see [1], [2], and [3]). In particular, it follows from the law of large numbers that a MGM portfolio will almost surely outperform other portfolios in the long run under mild conditions.

From a theoretic perspective, the "clear preference" for the MGM strategy is problematic since it does not maximize expected utility across all nondecreasing utility functions, even in the long run [4]. Levy [5] attempted to address this issue by showing that the expected utility of the MGM portfolio is no less than the expected utility of all other portfolios in the long run when the following conditions hold: (i) terminal wealth of portfolios are log-normally distributed, (ii) geometric standard deviation of the MGM portfolio is no less than geometric standard deviation of the other portfolios and (iii) marginal utility is bounded.

We now look into these three conditions:

- It follows from the central limit theorem that terminal wealth across an infinitely long investment horizon is log-normally distributed under mild conditions. Furthermore, this assumption appears to be reasonable under a sufficiently long investment horizon in practice. For example, based on the annual rates of returns of various assets from 1926 to 2012, Levy [5] observed that deviations between the log-normal distribution and empirical distributions based on actual returns appear negligible across an investment horizon of 20 years or longer.
- The requirement for higher geometric standard deviation is problematic as the terminal wealth of the MGM portfolio will almost surely be greater than the terminal wealth of all other portfolios, and not just those with smaller geometric standard deviation, in the long run. The argument presented by [5] does not explain why the MGM portfolio is also preferred by most investors over portfolios with larger geometric standard deviation in the long run.
- Losing one's entire wealth has very severe implications and it is often assumed that marginal utility is unbounded at zero wealth. Levy [5] suggested that marginal utility could be assumed to be bounded in practice because most investors do not allocate their entire wealth to investments. Hence, losing the entire investment capital is not equivalent to losing one's entire wealth. However, this does not mean that the investor's marginal utility

is bounded. Rather, the investor merely considers investment strategies that avoid zero wealth (e.g., no short selling).

In this paper, we show that the MGM strategy is clearly preferred in the long run from the perspective of log-weighted almost stochastic dominance. In particular, the MGM strategy is preferred by all investors whose utility function deviates marginally from the logarithm utility function. Furthermore, the maximum allowable deviation increases in the investment horizon and is unbounded. Unlike [5], we do not restrict ourselves to portfolios with smaller geometric standard deviation and allow for unbounded marginal utility.

2. Problem description

Let X_t denote the portfolio return (end of period value) in period t. For an investment horizon of T periods, the terminal wealth of the portfolio is $W_X(T) = \Pi_{t=1}^{T} X_t$. If each X_t is independent and follows a log-normal distribution with parameters μ_X and σ_X^2, $W_X(T)$ also follows a log-normal distribution with parameters $T\mu_X$ and $T\sigma_X^2$. Consider a second portfolio with terminal wealth $W_Y(T)$ that is log-normally distributed with parameters $T\mu_Y$ and $T\sigma_Y^2$.

Assuming that μ_X, μ_Y, σ_X and σ_Y are finite, it follows from the law of large numbers that the terminal wealth of the portfolio with higher geometric mean will almost surely be greater (see [5] for details). Stated formally, if $\mu_X > \mu_Y$ then:

$$P[W_X(T) > W_Y(T)] \to 1, \text{as}\, T \to \infty. \tag{1}$$

Here, we note that the assumption on log-normally distributed returns is not particularly restrictive since it follows from the central limit theorem that the terminal wealth distribution of both portfolios approach the log-normal distribution in the long run, even if X_t and Y_t are not log-normally distributed.

Although the argument above appears compelling, Merton and Samuelson [4] noted that the preferred investment strategy of an investor with iso-elastic utility is independent of the investment horizon and the MGM strategy does not maximize the expected utility of investors under some iso-elastic utility functions. Therefore, the MGM strategy does not dominate under conventional stochastic dominance rules in the long run and it is not immediately apparent that the MGM strategy should be preferred.

3. Weighted almost stochastic dominance

Stochastic dominance is a set of conditions that can be used to compare random variables when the utility function of the decision maker is unknown. For example, first-order stochastic dominance (FSD) is necessary and sufficient for higher expected utility across all nondecreasing utility functions. In practice, preference between random variables can be obvious in the absence of FSD (i.e., we do not need to account for all nondecreasing utility functions). This is illustrated in the St. Petersburg paradox [6], where the decision maker is asked to choose between a finite non-risky reward versus a risky reward with unbounded expected gains. Preference for the former is unanimous in practice even though it does not dominate the latter with FSD. In particular, Tan [7] noted that conventional stochastic dominance conditions, including FSD and higher-order stochastic dominance, account for linear utility functions but individuals are rarely completely risk-neutral in practice.

In response, Tan [7] proposed the concept of weighted almost stochastic dominance (WASD). Consider two random variables X and Y. Let U denote the set of all differentiable nondecreasing utility functions and $U^*(m(t),\varepsilon)$ denote the set of all differentiable nondecreasing utility functions whose marginal utility differs from $m(t)$ by a maximum factor of $\left[\frac{1}{\varepsilon}-1\right]^{0.5}$:

$$U^*(m(t),\varepsilon) = \left\{ u \in U : \left[\frac{1}{\varepsilon}-1\right]^{-0.5} m(t) \le u'(t) \le \left[\frac{1}{\varepsilon}-1\right]^{0.5} m(t), \forall t \right\}. \tag{2}$$

Definition 1 (Weighted Almost Stochastic Dominance)**.** *We say that X dominates Y with $(m(t),\varepsilon)$-WASD for some nonnegative function m and constant $\varepsilon \in (0,0.5]$ if and only if $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for all u in $U^*(m(t),\varepsilon)$.*

Tan [7] showed that, in the St. Petersburg paradox, a sufficiently large non-risky reward dominates the risky reward with $\left(\frac{1}{t},\varepsilon\right)$-WASD. For example, the expected utility of the risky reward is no greater than the utility associated with a non-risky reward of \$256 for all $u \in U^*(\frac{1}{t},0.001)$, the set of utility functions whose marginal utility deviates from $\frac{1}{t}$ (i.e., marginal utility of logarithm utility function) by a maximum factor of $\left[\frac{1}{0.001}-1\right]^{0.5} = 31.6$. Furthermore, ε

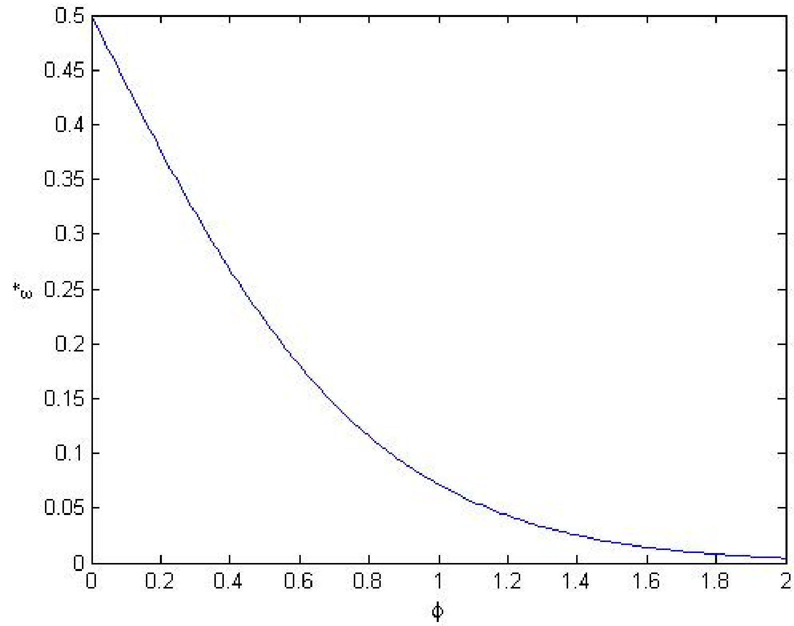

Figure 1: The relationship between ε^* and ϕ.

decreases as the non-risky reward increases. Therefore, preference for the non-risky reward becomes increasingly evident as the non-risky reward increases.

Weighted almost stochastic dominance between log-normally distributed random variables was recently discussed in [8]. Consider two random variables X and Y that are log-normally distributed with parameters μ_X, σ_X^2, μ_Y and σ_Y^2.

Theorem 1. *Suppose* $X \sim \ln N(\mu_X, \sigma_X^2)$, $Y \sim \ln N(\mu_Y, \sigma_Y^2)$, $\mu_X > \mu_Y$ *and* $\sigma_X \neq \sigma_Y$. *Let* $\phi = \frac{\mu_X - \mu_Y}{|\sigma_X - \sigma_Y|}$. *X dominates Y with* $\left(\frac{1}{t}, \varepsilon^*\right)$*-WASD where:*

$$\varepsilon^* = \frac{1}{2}\left[1 - \frac{\phi\sqrt{\pi}}{\phi\sqrt{\pi}\cdot erf(\frac{\phi}{\sqrt{2}}) + \sqrt{2}e^{-\frac{\phi^2}{2}}}\right], \tag{3}$$

and $erf(\cdot)$ *denotes the Gauss error function.*

Proof of Theorem 1. See [8].

Theorem 1 highlights that a log-normally distributed random variable with higher geometric mean dominates the other log-normally distributed random variable with $\left(\frac{1}{t}, \varepsilon^*\right)$-WASD. The corresponding values of ε^* for different ϕ are illustrated in Figure 1.

4. Dominance of MGM portfolio

We begin this section with the following two propositions.

Proposition 1. ε^* *is strictly decreasing in* ϕ*.*

Proof of Proposition 1. We take the derivative of ε^*.

$$\begin{aligned}
\frac{d\varepsilon^*}{d\phi} &= -\frac{1}{2}\cdot\frac{\sqrt{\pi}\left(\phi\sqrt{\pi}\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2}e^{-\frac{\phi^2}{2}}\right)}{\left(\phi\sqrt{\pi}\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2}e^{-\frac{\phi^2}{2}}\right)^2} \\
&\quad -\frac{1}{2}\cdot\frac{-\phi\sqrt{\pi}\left(\sqrt{\pi}\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2}\phi e^{-\frac{\phi^2}{2}}-\sqrt{2}\phi e^{-\frac{\phi^2}{2}}\right)}{\left(\phi\sqrt{\pi}\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2}e^{-\frac{\phi^2}{2}}\right)^2} \qquad (4)\\
&= -\frac{1}{2}\cdot\frac{\pi\phi\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2\pi}e^{-\frac{\phi^2}{2}}-\pi\phi\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)}{\left(\phi\sqrt{\pi}\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2}e^{-\frac{\phi^2}{2}}\right)^2} \qquad (5)\\
&= -\frac{1}{2}\cdot\frac{\sqrt{2\pi}e^{-\frac{\phi^2}{2}}}{\left(\phi\sqrt{\pi}\cdot erf\left(\frac{\phi}{\sqrt{2}}\right)+\sqrt{2}e^{-\frac{\phi^2}{2}}\right)^2} \qquad (6)\\
&< 0. \quad \square
\end{aligned}$$

Proposition 2. $\lim_{\phi\to\infty}\varepsilon^* = 0$.

Proof of Proposition 2. Follows from the observation that

$$\lim_{\phi\to\infty} erf\left(\frac{\phi}{\sqrt{2}}\right) = 1 \qquad (7)$$

and

$$\lim_{\phi\to\infty}\phi^{-1}e^{-\frac{\phi^2}{2}} = 0. \quad \square \qquad (8)$$

Proposition 1 highlights that the set of utility functions where preference for the portfolio with higher geometric mean is guaranteed increases with ϕ (i.e., ε^* decreases in ϕ). Proposition 2 highlights that the increase is unbounded (i.e., $\varepsilon^* \to 0$). Having proven these results, we are now ready to show the dominance of the MGM portfolio in the long run.

Let $W_X(T)$ denote the terminal wealth of the MGM portfolio at period T where $W_X(T)$ is log-normally distributed with parameters $T\mu_X$ and $T\sigma_X^2$. In addition, let $W_Y(T)$ denote the terminal wealth of another portfolio at period T where $W_Y(T)$ is log-normally distributed with parameters $T\mu_Y$ and $T\sigma_Y^2$. Since the MGM portfolio has maximal geometric mean, $\mu_X > \mu_Y$.

If $\sigma_X = \sigma_Y$, preference for $W_X(T)$ over $W_Y(T)$ is apparent as the former dominates the latter with FSD [9]. If $\sigma_X \neq \sigma_Y$, it follows from Theorem 1 that $W_X(T)$ dominates $W_Y(T)$ with $\left(\frac{1}{t},\varepsilon^*\right)$-WASD, where:

$$\phi = \frac{T\mu_X - T\mu_Y}{\left|\sqrt{T}\sigma_X - \sqrt{T}\sigma_Y\right|} = \frac{\sqrt{T}(\mu_X-\mu_Y)}{|\sigma_X-\sigma_Y|}. \qquad (9)$$

Since ϕ increases with T, it follows from Proposition 1 that the maximum allowable deviation from logarithm utility, such that $W_X(T)$ is clearly preferred, increases with T. Therefore, preference for the MGM portfolio becomes clearer as the investor's investment horizon increases. Furthermore, since $\phi \to \infty$ as $T \to \infty$, it follows from Proposition 2 that the maximum allowable tolerance is unbounded. Hence, the maximum allowable tolerance from logarithm utility becomes infinitely large as the investment horizon grows infinitely long.

From the perspective of $\left(\frac{1}{t},\varepsilon\right)$-WASD, preference for the MGM portfolio becomes increasingly obvious as the investment horizon increases, which is consistent with the observation that a portfolio with higher geometric mean will almost surely have higher terminal wealth in the long run.

5. Conclusions

In this paper, we address the gap between the preference for the MGM strategy in the long run from the perspectives of the law of large numbers and stochastic dominance. In particular, the former states that the MGM strategy is almost surely to be better in the long run but preference for the MGM strategy is indistinct under conventional stochastic dominance rules (i.e., MGM portfolio is not preferred under some utility functions). Here, we explain why the clear preference for the MGM strategy in the long run can be explained via log-weighted almost stochastic dominance.

Besides resolving the theoretic debate regarding the preference of the MGM strategy in the long run, this work also adds to the stochastic dominance literature by providing additional support for the use of log-weighted almost stochastic dominance (i.e., consider all $u \in U^*(m(t), \varepsilon)$) to explain clear preferences between risky prospects in practice. Previously, Tan [7] has shown that log-weighted almost stochastic dominance can reveal the clear preference for the non-risky reward in the St. Petersburg paradox. In this paper, we show that log-weighted almost stochastic dominance can also reveal the clear preference for the MGM strategy in the long run.

Finally, this work presents an alternative to the mean-variance framework proposed by [10], which is often criticized for assumptions on normality and quadratic utility. In our work, we highlight that a geometric-mean-geometric-standard-deviation framework is suitable for comparing between investments with log-normal returns, which is reasonable for investors with a sufficiently long investment horizon. Here, we do not assume that the utility of the decision maker follows any particular form but only assume that it can be approximated by logarithm utility. One key insight is that the greater the difference in geometric mean and the smaller the difference in geometric standard deviation, the clearer the preference for the investment with higher geometric mean.

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References

[1] Kelly, J. L., 1956, "A New Interpretation of Information Rate," System Tech Journal, 35(4), 917-926.

[2] Breiman, L., 1960, "Investment Policies for Expanding Businesses Optimal in A Long-run Sense," Naval Research Logistics Quarterly, 7(4), 647-651.

[3] Markowitz, H., 1976, "Investment for the Long Run: New Evidence for An Old Rule," The Journal of Finance, 31(5), 1273-1286.

[4] Merton, R. C. and Samuelson, P. A., 1974, "Fallacy of the Log-normal Approximation to Optimal Portfolio Decision-making Over Many Periods," Journal of Financial Economics, 1(1), 67-94.

[5] Levy, H., 2016, "Aging Population, Retirement, and Risk Taking," Management Science, 62(5), 1415-1430.

[6] Bernoulli, D., 1738, "Specimen theoriae novae de mensura sortis," Commentarii Academiae Scientiarum Imperialis Petropolitanae. Translated, Bernoulli, D., 1954, "Exposition of a new theory on the measurement of risk," Econometrica, 22, 23-36.

[7] Tan, C. H., 2015, "Weighted Almost Stochastic Dominance: Revealing the Preferences of Most Decision Makers in the St. Petersburg Paradox," Decision Analysis, 12(2), 74-80.

[8] Yang, Z., 2016, "Optimization for Decision Makers with Ambiguous Preferences," Ph.D. dissertation, National University of Singapore.

[9] Levy, H., 1973, "Stochastic Dominance Among Log-normal Prospects," International Economic Review, 14(3), 601-614.

[10] Markowitz, H., 1952, "Portfolio Selection," The Journal of Finance, 7(1), 77-91.